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Local Best Rational Approximations to Continuous Functions and the Rays They Emanate

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Let f be a continuous real function defined on $[0, 1]$. A real rational function $r_0 \in R_n^m(\mathbb{C})$ is a local best approximation to $cf + (1-c)\, r_0$ for each $c > 0$ if and only if r_0 is a global best approximation to f from Re $R_n^m(\mathbb{C})$.

I. INTRODUCTION

Suppose that a real rational function r_0 is a local best approximation to a continuous real function f from the real rational functions R_n^m. It is known then that r_0 is a global best approximation to f, and that it also is a best approximation to each function on the ray $\{f_c = cf + (1-c)\, r_0 : c \geqslant 0\}$. However, if r_0 is the best approximation from $R_n^m(\mathbb{C})$—the complex valued rationals defined on the unit interval—it is not necessarily a best approximation to each f_c. Moreover it is not known, in the complex setting, if r_0 being a local best approximation implies that it is a global best approximation.

We show here that r_0 being a local best approximation to all f_c from $R_n^m(\mathbb{C})$ is a very strong condition, equivalent to r_0 being a global best approximation from Re $R_n^m(\mathbb{C})$.

Notation. The real polynomials of degree less then or equal to k which are defined on $[0, 1]$ are denoted by $\mathscr{P}_k$. The corresponding complex polynomials are written $\mathscr{P}_k(\mathbb{C})$. The degree of a polynomial p is ∂p.

$$\mathscr{P}_k^+ = \{p \in \mathscr{P}_k : p(x) \neq 0 \text{ for } 0 \leqslant x \leqslant 1\}, \tag{1.1}$$

and

$$\mathscr{R}_n^m = \{p/q : p \in \mathscr{P}_m,\ q \in \mathscr{P}_n^+\}. \tag{1.2}$$

Analagous statements define $\mathscr{P}_k^+(\mathbb{C})$ and $\mathscr{R}_n^m(\mathbb{C})$.

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For a function g and a set $K \subseteq [0, 1]$,

$$\|g\|_K = \sup\{|g(k)|: k \in K\}, \tag{1.3}$$

and

$$\|g\| = \|g\|_{[0,1]}. \tag{1.4}$$

We use

$$\text{crit } g = \{x \in [0, 1]: |g(x)| = \|g\|\}, \tag{1.5}$$

and

$$\text{sgn } g = \begin{cases} \dfrac{g(x)}{|g(x)|}, & x \neq 0 \\ 0, & x = 0. \end{cases} \tag{1.6}$$

As usual Re g and Im g represent the real and imaginary parts of g.

For a set, A, of functions on $[0, 1]$,

$$\begin{aligned} \text{Re } A &= \{\text{Re } g: g \in A\} \\ \text{Im } A &= \{\text{Im } g: g \in A\}. \end{aligned} \tag{1.7}$$

A function f is said to have $g \in A$ as a *best approximation* from A if

$$\|f - g\| = \inf\{\|f - a\|: a \in A\}. \tag{1.8}$$

If there is a neighborhood U of g such that g is a best approximation to f from $A \cap U$, than g is a *local best approximation* to f.

Reserved Notation. We will reserve the following notation throughout the paper, $p_0 \in \mathscr{P}_m$, $q_0 \in \mathscr{P}_n$. We assume that p_0 and q_0 have no common factors.

$$r_0 = p_0/q_0, \tag{1.9}$$

and

$$d = \max\{m - \partial p_0, n - \partial q_0\}. \tag{1.10}$$

Let f be a continuous real function on $[0, 1]$; we write, for c real,

$$f_c = cf + (1 - c)\, r_0, \tag{1.11}$$

$$e_c = f_c - r_0 \qquad \text{and} \qquad e = e_1. \tag{1.12}$$

II. Estimates for $\|e_c\|$

The proof of the main theorem uses numerous computations. This section collects results which conclude that a function g has the property that $\|e_c - g\| < \|e_c\|$.

LEMMA 2.1. *If*

$$\|e - g\|_{\text{crit } e} < \|e\|_{\text{crit } e},$$

then for large c,

$$\|e_c - g\| < \|e_c\|.$$

Proof. There is a neighborhood U of crit e for which

$$\|e - g\|_U < \|e\|. \tag{2.1}$$

Hence

$$\begin{aligned} \|e_c - g\|_U &\leqslant \|(c-1)(f - r_0)\|_U + \|f - r_0 - g\|_U \\ &\leqslant (c-1)\, \|e\|_U + \|e\| \\ &\leqslant c\, \|e\| \\ &= \|e_c\|. \end{aligned} \tag{2.2}$$

For points not in U we have an $\varepsilon > 0$ such that

$$\|e\|_{[0,1]-U} \leqslant \|e\| - \varepsilon. \tag{2.3}$$

So in this case,

$$\begin{aligned} \|e_c - g\|_{[0,1]-U} &\leqslant \|e_c\|_{[0,1]-U} + \|g\| \\ &\leqslant c[\|e\| - \varepsilon] + \|g\|. \end{aligned} \tag{2.4}$$

So if

$$\begin{gathered} c > \|g\|/\varepsilon, \\ \|e_c - g\|_{[0,1]-U} \leqslant c\, \|e\| = \|e_c\|. \end{gathered} \tag{2.5}$$

Combining (2.3) and (2.5) proves the lemma. ∎

LEMMA 2.2. *If*

$$\|e - \text{Re } g\| < \|e\|,$$

then for large c

$$\|e_c - g\| < \|e_c\|.$$

Proof. From Lemma 2.1 we need only show that

$$\|e - g\|_{\text{crit } e} < \|e\|. \tag{2.6}$$

For x in crit $e = $ crit e_c,

$$|(e_c - g)(x)|^2 < \|e_c\|^2, \tag{2.7}$$

if and only if

$$\|e_c\|^2 - 2ce(x) \operatorname{Re} g(x) + [\operatorname{Re} g(x)]^2 + [\operatorname{Im} g(x)]^2 \leqslant \|e_c\|^2, \tag{2.8}$$

if and only if

$$|(e - g)(x)|^2 - (c - 1)\, e(x) \operatorname{Re} g(x) + [\operatorname{Im} g(x)]^2 \leqslant \|e\|^2, \tag{2.9}$$

if and only if

$$[\operatorname{Im} g(x)]^2 \leqslant (c - 1)\, e(x). \tag{2.10}$$

From the hypothesis, Re $g(x)$ must be a nonzero number of the same sign as $e(x)$. Therefore the right side of the inequality can be made arbitrarily large with c. ∎

LEMMA 2.3. *If g is a real valued function such that* sgn$[g(x)] =$ sgn$[e(x)]$ *for all x in* crit e, *then for all sufficiently large c,*

$$\|e_c - g\| < \|e_c\|.$$

Proof. If $\|g\|_{\text{crit } e} < \|e_c\|$, then $\|e_c - g\|_{\text{crit } e} < \|e_c\|$. Hence the lemma follows from Lemma 2.1. ∎

LEMMA 2.4. $x \in$ crit e, *then*

$$\|e\|^2 - |e(x) - g(x)|^2 = 2e(x) \operatorname{Re} g(x) - |g(x)|^2.$$

Proof. This is acquired by just expanding $|e(x) - g(x)|^2$. ∎

III. CLASSES OF POLYNOMIALS

LEMMA 3.1.

$$P_0 \mathscr{P}_n + q_0 \mathscr{P}_m = \mathscr{P}_{m+n-d}.$$

Proof. A proof for this known result can be built using the degrees of the polynomials and the dimensions of the linear spaces. (For example, see [1].) ∎

Notation. For a complex j times differential function f put

$$Z(f)=\{\omega\in\mathbb{C}: f(\omega)=0\},$$

$$Z_2(f)=\{\omega\in Z(f): f'(\omega)=0\},$$

and

$$Z_j(f)=\{\omega\in Z_{j-1}(f): f^{(j)}(\omega)=0\}.$$

LEMMA 3.2.

> $\{\gamma^2+q_0\mathscr{P}_n: \gamma\in\mathscr{P}_n\}\supset\{t\in\mathscr{P}_{2n}$: (1) $Z(t)\cap Z_2(q_0)=\varnothing$, (2) *If* $\partial t>n+\partial q_0$, *then t is even and t has a positive leading coefficient, and* (3) $t\geqslant 0$ *on* $Z(q_0)\cap R\}$.

Proof. We wish to find a $\gamma\in\mathscr{P}_n$ so that γ^2 agrees with t on the zeros of q_0, and which has coefficients that agree with those of t for powers of x greater than $n+\partial q_0$. For such a γ, $t-\gamma^2$ is a polynomial of degree $n+\partial q_0$ which has q_0 as a factor and the lemma will be proven.

Let $H\in\mathscr{P}_{\partial q_0}$ be chosen so that H^2 agrees with t on the zeros of q_0—including multiple zeros. For example, on a double zero of q_0, the derivative of H^2 agrees with that of t. This Hermite-type interpolation is possible since H is not zero on a multiple zero of q_0. (We use conditions (1) and (3) hypothesised for t in defining H.)

We now wish to find $S\in\mathscr{P}_{n-\partial q_0}$ so that

$$(Sq_0+H)^2=\gamma^2 \tag{3.1}$$

has coefficients of $X^{n+\partial q_0+1}$, $X^{n+\partial q+2}$, ..., X^{2n} that agree with those of t. If $It\leqslant n+\partial q_0$ this is satisfied with S equal zero. Hence we will assume that $\partial t=2k>n+\partial q_0$. We proceed by examining the coefficients in the expansion of $(Sq_0+H)^2$ (the coefficients of q_0 and H are already fixed: those for S are to be determined). For j larger than $k-\partial q_0$ put the coefficient of x^j for S equal to zero.

The leading coefficient of $(Sq_0+H)^2$ is the product of the squares of the leading coefficients of S and q_0. Since the leading coefficient of t is positive, the coefficient of $x^{k-\partial q_0}$ for S is determined. If $0<j<k-\partial q_0$, the coefficient of x^{2k-j} in the expansion of $(Sq_0+H)^2$ can be written as the sum of two terms. One consists of twice the product of the lead coefficients of S, q_0, and s_i the coefficient of $x^{\partial s-i}$ in S. The other is an expression composed of coefficients of q_0 and already determined coefficients of S. Hence s_j can be chosen so that the coefficients of x^{2k-j} in t and $(Sq_0+H)^2$ are equal. ∎

COROLLARY 3.3.

$$\{\gamma^2 + q_0\mathscr{P}_n : \gamma \in \mathscr{P}_n\} \supseteq \{q^2 + s^2 : q, s \in \mathscr{P}_n, Z(q^2 + s^2) \cap Z(q_0) = \varnothing\}$$

LEMMA 3.4.

$$\{\omega(p_0\beta - q_0\alpha)\, q_0 - p_0\gamma^2 + q_0\gamma\delta : \omega \in \mathscr{P}_d, \alpha, \delta \in \mathscr{P}_m, \beta, \gamma \in \mathscr{P}_n\}$$
$$\supseteq \{\mathscr{P}_{m+n}q_0 - p_0(q^2 + s^2) : q, s \in \mathscr{P}_n, \text{ and } Z(y^2 + s^2) \cap Z(q_0) = \varnothing\}.$$

Proof. From Corollary 3.3, each member of this set can be written in the form $q_0 v - p_0\gamma^2$ for some $v \in \mathscr{P}_{m+n}$. Choose δ so that $v - \gamma\delta$ has a real zero. Then there are $\omega \in \mathscr{P}_d$ and $u \in \mathscr{P}_{m+n-d}$ such that

$$\omega u = v - \gamma\delta. \tag{3.2}$$

Now choose α and β so that

$$p_0\beta - q_0\alpha = u. \tag{3.3}$$

We have

$$\begin{aligned} \omega(p_0\beta - q_0\alpha)\, q_0 - p_0\gamma^2 + q_0\gamma\delta &= \omega u q_0 - p_0\gamma^2 + q_0\gamma\delta \\ &= v q_0 - \gamma\delta q_0 - p_0\gamma^2 + q_0\gamma\delta \\ &= v q_0 - p_0\gamma^2. \quad \blacksquare \end{aligned} \tag{3.4}$$

IV. NOTATIONAL CONVENTION

Let $\omega \in d$, $\alpha, \delta \in \mathscr{P}_m$, and $\beta, \gamma \in \mathscr{P}_n$. For the remainder of the paper we will write, for λ real,

$$r_\lambda = \frac{\omega p_0 + \lambda^2\alpha + i\lambda\delta}{\omega q_0 + \lambda^2\beta + i\lambda\gamma} \tag{4.1}$$

$$r = r_1 \tag{4.2}$$

$$L_\lambda = \frac{\lambda^2[\beta p_0 - \alpha q_0]\, \omega q_0 - \gamma^2 p_0 + \gamma\delta q_0 + i\lambda[\gamma p_0 - \delta q_0]\, \omega q_0}{q_0^3\omega^2} \tag{4.3}$$

and

$$L = L_1. \tag{4.4}$$

V. Local Best Approximations

The next two lemmas record the result of straightforward computation from definitions.

Lemma 5.1.

$$r_0 - r_\lambda = \lambda^2 \left\{ \frac{[\beta p_0 - \alpha q_0][\omega q_0 + \lambda^2 \beta] - [\gamma^2 p_0 - \delta\gamma q_0]}{q_0[(\omega q_0 + \lambda^2\beta)^2 + (\lambda\gamma)^2]} \right\}$$
$$+ i\lambda \left\{ \frac{(\gamma p_0 - \delta q_0)(\omega q_0 + \lambda^2\beta) - \lambda^2\gamma[\beta p_0 - \alpha q_0]}{q_0[(\omega q_0 + \lambda^2\beta)^2 + (\lambda\gamma)^2]} \right\}.$$

Lemma 5.2.

$$\lim_{\lambda \to 0} \left\| \frac{r_0 - r_\lambda - L_\lambda}{\lambda^2} \right\| = 0.$$

Lemma 5.3. *If*

$$\|e - L\|_{\text{crit } e} < \|e\|,$$

then for all sufficiently small λ

$$\|e - r_\lambda\| < \|e\|.$$

Proof. From Lemma 2.4 there is an $\varepsilon > 0$ such that on crit e

$$2e[\operatorname{Re} L] > |L|^2 + \varepsilon. \tag{5.1}$$

This inequality must also hold on some neighborhood U of crit e. It is also true that on U for $0 < \lambda \leqslant 1$,

$$2e\,\lambda^2 \operatorname{Re} L > \lambda^4[\operatorname{Re} L]^2 + \lambda^2[\operatorname{Im} L]^2 + \lambda^2\varepsilon. \tag{5.2}$$

It is always true (because e is real) that

$$|e - L_\lambda|^2 \leqslant \|e\|^2 - 2e \operatorname{Re} L_\lambda + |L_\lambda|^2. \tag{5.3}$$

Since

$$\lambda^2 \operatorname{Re} L = \operatorname{Re} L_\lambda \qquad \text{and} \qquad \lambda \operatorname{Im} L = \operatorname{Im} L_\lambda, \tag{5.4}$$

line (5.2) shows that on U,

$$|e - L_\lambda|^2 \leqslant \|e\|^2 - \lambda^2\varepsilon$$
$$\leqslant \left[\|e\| - \lambda^2 \frac{\varepsilon}{2\,\|e\|} \right]^2. \tag{5.5}$$

From Lemma 5.2 we conclude that for small positive λ

$$\|f - r_\lambda\|_U < \|e\|. \tag{5.6}$$

Since r_λ converges uniformly to r_0, and since there is a positive μ for which

$$\|e\|_{[0,1]-U} < \|e\| - \mu, \tag{5.7}$$

we obtain that for all sufficiently small λ,

$$\|f - r_\lambda\|_{[0,1]-U} < \|e\|. \tag{5.8}$$

This combines with line (5.6) to prove the lemma. ∎

THEOREM. *If r_0 is a local best approximation to f_c for all $c > 0$ then r_0 is a global best approximation to f from* Re $R_n^m(\mathbb{C})$.

Proof. This is now just a matter of piecing together the previous lemmas. Suppose there is a function

$$\rho = \frac{p + it}{q + is} \in R_n^m(\mathbb{C}) \tag{5.9}$$

for which

$$\|f - \operatorname{Re} \rho\| < \|e\|. \tag{5.10}$$

We may also assume that $Z(q^2 + s^2) \cap Z(q_0) = \varnothing$. From Lemma 3.4 choose ω, α, β, δ, and γ so that

$$\omega(p_0\beta - q_0\alpha)\, q_0 - p_0\gamma^2 + q_0\delta = q_0[pq - st] - p_0[q^2 + s^2], \tag{5.11}$$

and construct r, r_λ, L, and L_λ as in equations (4.1)–(4.4). By Lemma 2.3 we have that for a sufficiently large c

$$\|e_c - \operatorname{Re} L\| < \|e_c\|. \tag{5.12}$$

From Lemma 2.2 we can in fact assume

$$\|e_c - L\| < \|e_c\|. \tag{5.13}$$

Replacing f by f_c for some large c, we may assume that

$$\|e - L\| < \|e\|. \tag{5.14}$$

From Lemma 5.3, r_0 is not a local best approximation to f. ∎

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